Flows in a rotating frame

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Part III Preparatory Workshop 2020

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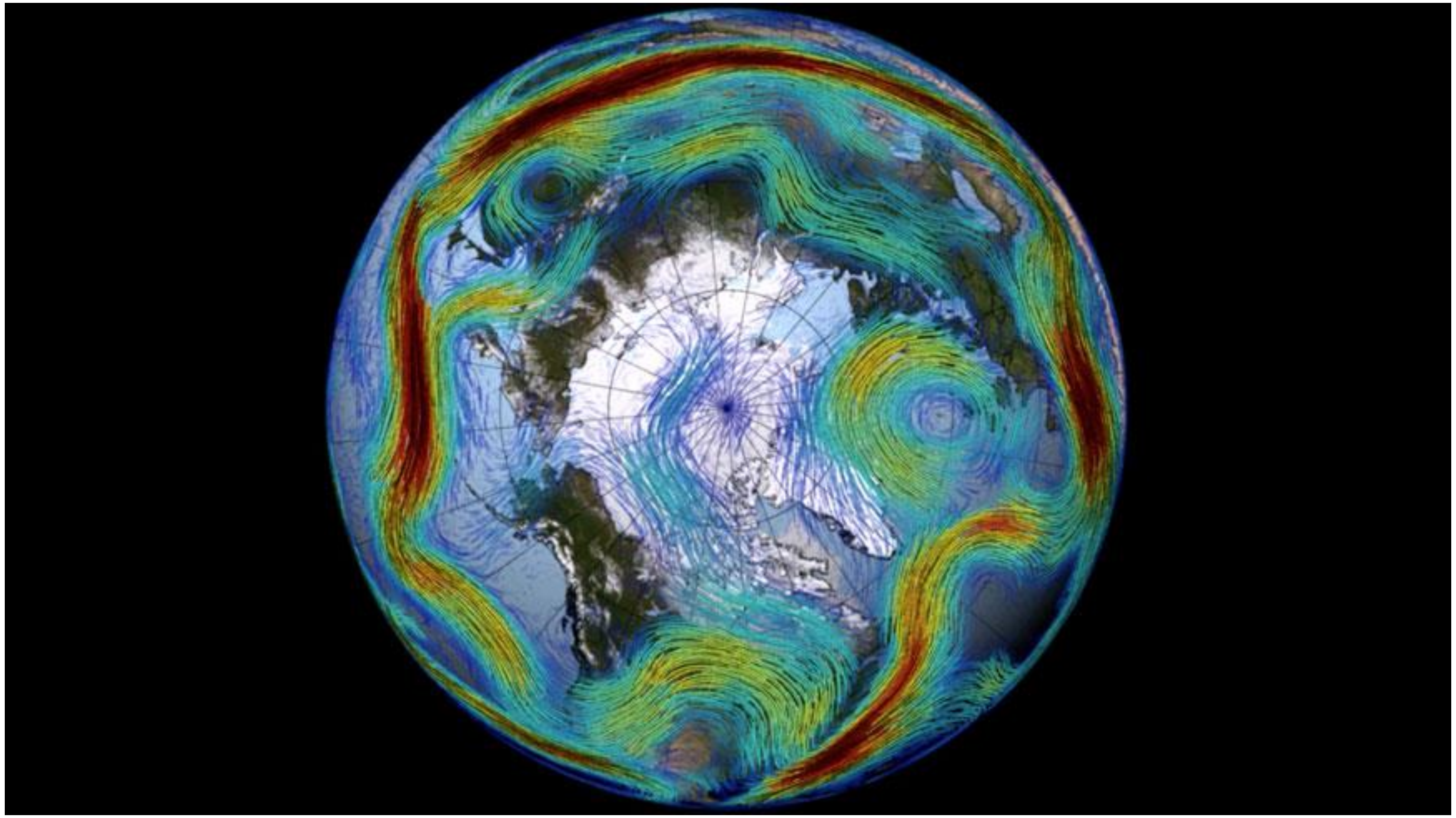
Motivation



National Ocean Service, NOAA, U.S. Department of Commerce

Equations of motion

In a rotating frame of reference

$$\rho\left(\frac{\partial \mathbf{u}}{\partial t} + \cancel{\mathbf{u}\cdot\nabla\mathbf{u}} + 2\boldsymbol{\Omega}\times\mathbf{u}\right) = -\nabla p + \cancel{\mu\nabla^2\mathbf{u}} - \cancel{\rho\boldsymbol{\Omega}\times(\boldsymbol{\Omega}\times\mathbf{x})} + \rho\mathbf{g}$$

negligible in ocean/atmosphere

$$\frac{|\underline{u}\cdot\nabla\underline{u}|}{|\underline{\Omega}\times\underline{u}|} \sim \frac{U}{\Omega L}$$

$$\frac{|\underline{\Omega}\times(\underline{\Omega}\times\underline{x})|}{|\underline{g}|} \sim 4\times10^{-3}$$

$|\underline{\Omega}| \sim \frac{2\pi}{\text{day}}$

$|\underline{x}| \sim 10^4$ km

$|\underline{g}| \sim 10$ m^{2}s^{-1}

$$\boxed{\mathrm{Ro} = \frac{U}{\Omega L}}$$ Rossby number

atmosphere: $U \sim 10$ ms^{-1}

$L \sim 10^3 - 10^4$ km $\Rightarrow$ Ro $\sim 10^{-1} - 10^{-2}$

$\Omega \sim 10^{-4}$ s^{-1}

Euler's equation in a rotating frame:

$$\boxed{\frac{\partial \mathbf{u}}{\partial t} + 2\boldsymbol{\Omega}\times\mathbf{u} = -\frac{1}{\rho}\nabla p + \mathbf{g}}$$

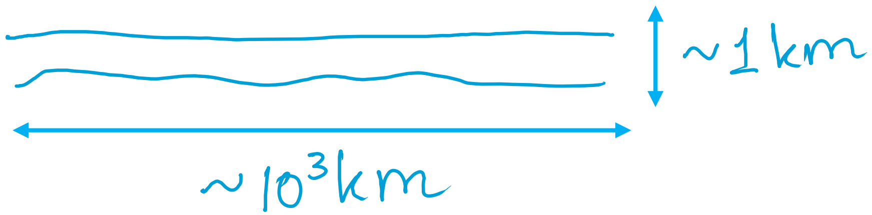
Coriolis parameter

$\underline{f} = 2\underline{\Omega}$

Coriolis parameter / Planetary vorticity

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{f} \times \mathbf{u} = -\frac{1}{\rho}\nabla p + \mathbf{g}$$

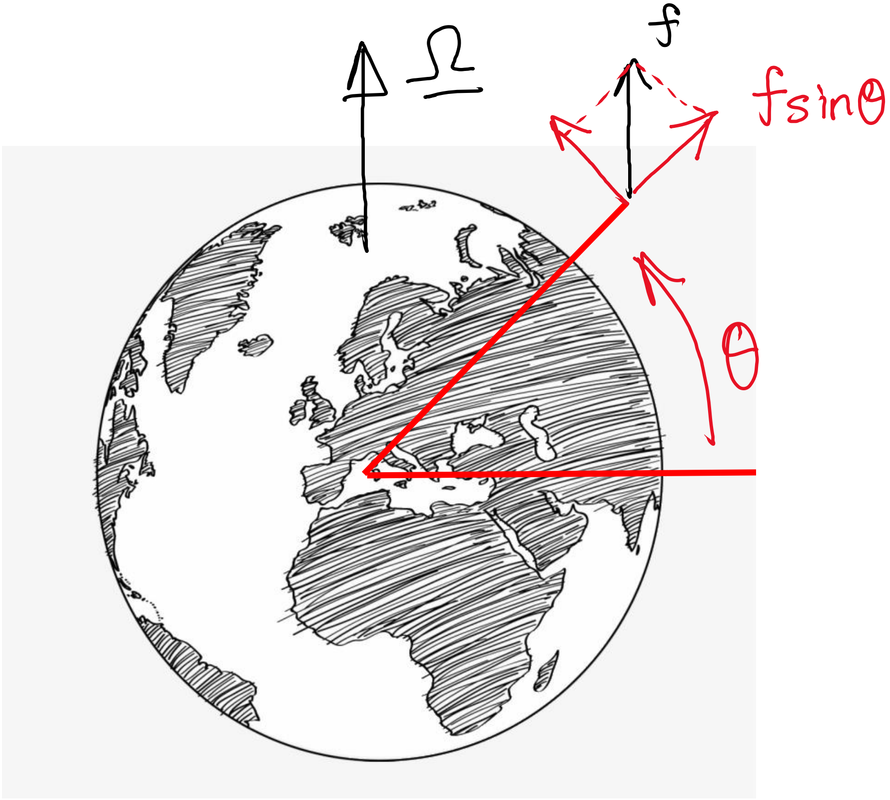
Ocean :



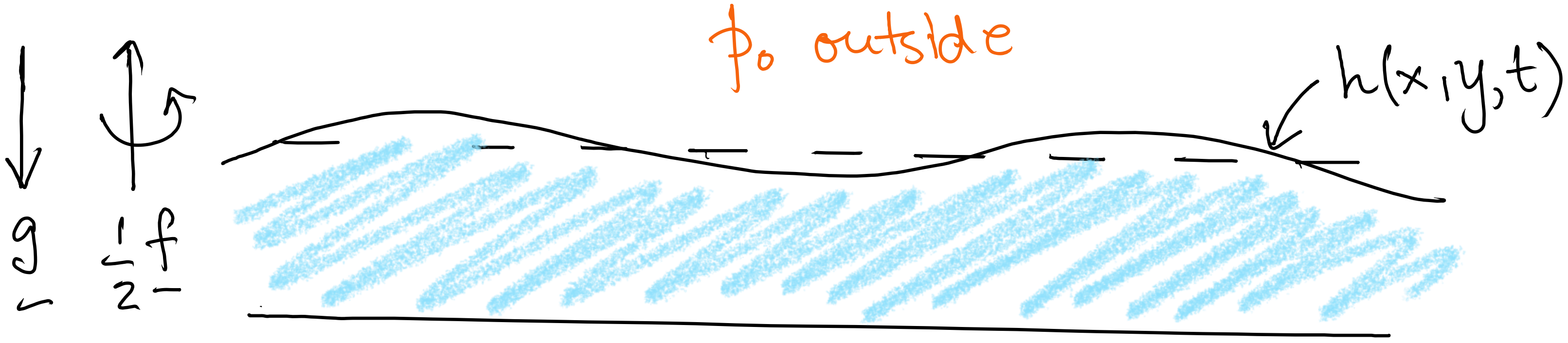
$\Rightarrow$ $\underline{u}$ mostly horizontal

$\Rightarrow$ only vertical component of $\underline{f}$ matters

$$\underline{f} \mapsto \underline{f} = 2\Omega \sin\theta \, \underline{e}_z \equiv f \underline{e}_z$$



Shallow water equations



Mostly horizontal

$\Rightarrow \underline{u} = (u, v, 0)$

$\underline{f} = (0, 0, f)$

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{f} \times \mathbf{u} = -\frac{1}{\rho}\nabla p + \mathbf{g}, \quad \mathbf{f} = f\mathbf{e}_z$$

$$\frac{\partial u}{\partial t} - fv = -\frac{1}{\rho}\frac{\partial p}{\partial x} = -g\frac{\partial h}{\partial x}$$

$$\frac{\partial v}{\partial t} + fu = -\frac{1}{\rho}\frac{\partial p}{\partial y} = -g\frac{\partial h}{\partial y}$$

$$0 = -\frac{1}{\rho}\frac{\partial p}{\partial z} - g \Rightarrow p = p_0 + \rho g(h - z)$$

Geostrophic balance

steady

$$\cancel{\frac{\partial u}{\partial t}} - fv = -g\frac{\partial h}{\partial x}$$

$$\cancel{\frac{\partial v}{\partial t}} + fu = -g\frac{\partial h}{\partial y}$$

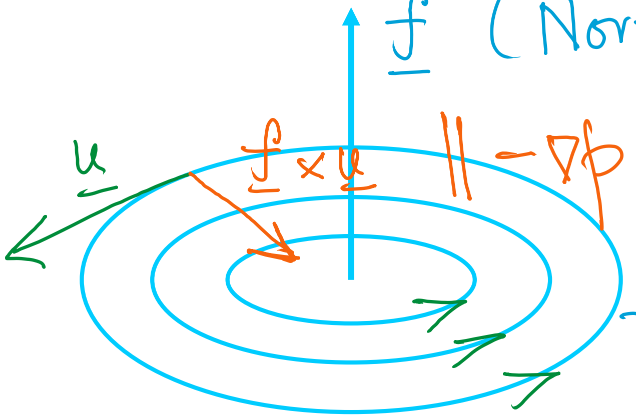
$\underline{f} \times \underline{u} = -\frac{1}{\rho}\nabla \phi$

$$u = \frac{\partial}{\partial y}\left(-\frac{gh}{f}\right), \quad v = -\frac{\partial}{\partial x}\left(-\frac{gh}{f}\right)$$

Looks like $u = \frac{\partial \psi}{\partial y}$, $v = -\frac{\partial \psi}{\partial x}$

$$\boxed{\psi = -\frac{gh}{f}}$$

Shallow water streamfunction



$\Rightarrow$ low h/ϕ in centre

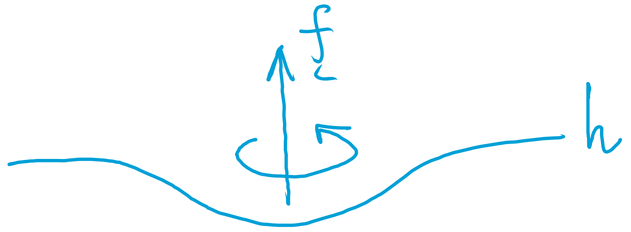
streamlines

contours of $h/\phi/\psi$

Cyclones L H

Potential vorticity

$$\mathbf{Q} = \nabla \times \mathbf{u} - \frac{h - h_0}{h_0}\mathbf{f}$$



Proposition. Potential vorticity is conserved,

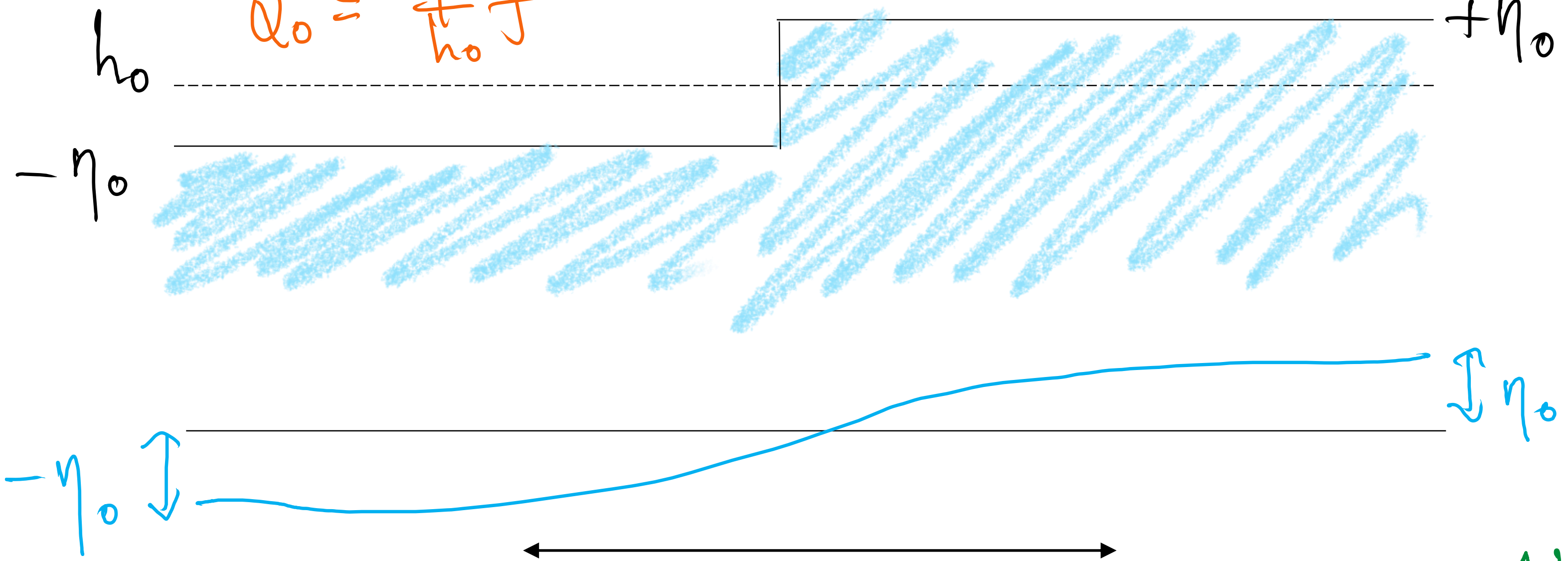
$$\frac{\partial \mathbf{Q}}{\partial t} = 0.$$

Proof. From conservation of mass and kinematic BC at sutface. See lecture notes linked in the PDF for mathematical detail.

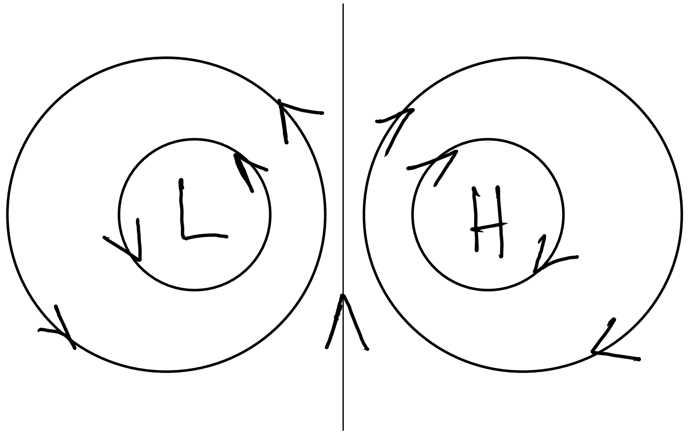
Example

$$\mathbf{Q} = \nabla \times \mathbf{u} - \frac{h - h_0}{h_0}\mathbf{f}$$

$Q_0 = \frac{\eta_0}{h_0} f$ $\qquad$ $Q_0 = -\frac{\eta_0}{h_0} f$



What is this lengthscale?



Atmosphere : $R \sim \frac{\sqrt{10 \cdot 10^3}}{10^{-4}} \approx 10^3 \text{km}$

Ocean : $R \sim \frac{\sqrt{10 \cdot 10}}{10^{-4}} \approx 100 \text{ km}$

<u>Dimensional analysis</u>: $[g] = m s^{-2}$ $\quad [f] = s^{-1}$ $\quad \Rightarrow \quad R = \frac{\sqrt{g h_0}}{f}$ Rossby radius of deformation

$[h_0] = m$

Final remarks

- Remember to attempt the exercises for this topic before the live session on

2pm Thursday, 8 October

- Look inside the PDF notes for a link to a video about *Rossby waves and extreme weather events*.
- If you find any typos/mistakes in the PDF notes, please email me at mt599@cam.ac.uk

Thank you for watching!