Stokes flow

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Part III Preparatory Workshop 2020

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What is Stokes flow?

also laminar/creeping flow

Stokes flow = negligible inertia/internal friction dominates

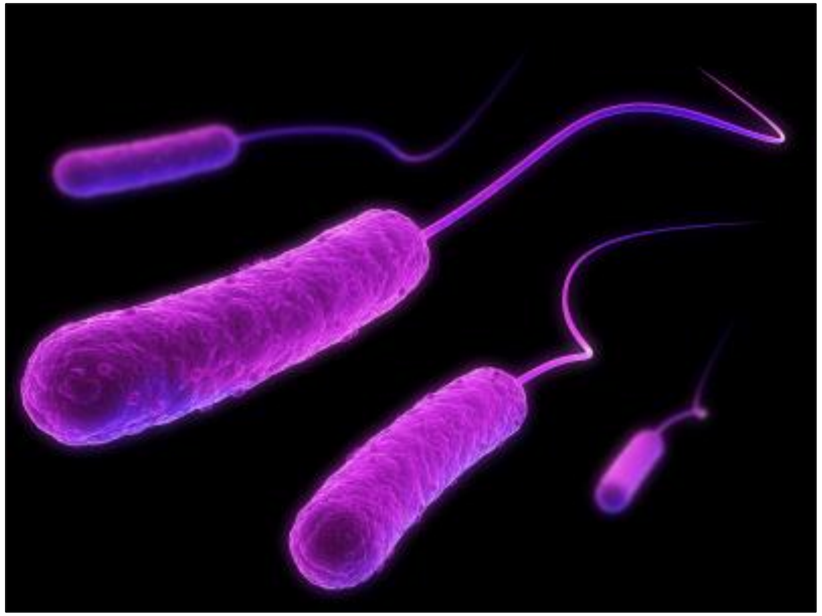
Very viscous

Source: pinterest.com

Very slow

Source: www.swisseduc.ch

Very small



Source: physics.aps.org

Stokes equations

$$\underbrace{\rho\left(\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u}\cdot\nabla\mathbf{u}\right)}_{\sim \frac{\rho U^2}{L} \text{ inertial effects}} = -\nabla p + \underbrace{\mu\nabla^2\mathbf{u}}_{\text{viscous effects } \sim \frac{\mu U}{L^2}} + \mathbf{f}.$$

Definition. Reynolds number, $\mathrm{Re} = \frac{\rho U L}{\mu}$ $\rightarrow$ $\nu = \frac{\mu}{\rho}$, kinematic viscosity

water

$\nu \approx 10^{-6}\,\mathrm{m}^2\mathrm{s}^{-1}$

$0 = \nabla\cdot\underline{u}$

In the limit Re ≪ 1

$$\boxed{\begin{aligned} 0 &= -\nabla p + \mu\nabla^2\mathbf{u} + \cancel{\mathbf{f}} \\ \nabla\cdot\mathbf{u} &= 0 \end{aligned}}$$

often, adsorb into ∇p

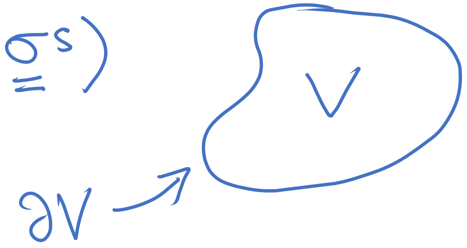
Stokes equations

Properties of Stokes flow ?

$$\boxed{\nabla \cdot \mathbf{u} = 0, \quad \nabla \cdot \sigma = 0}$$

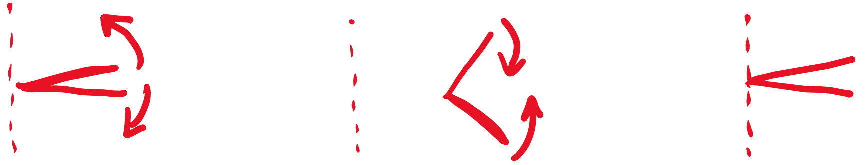
solution $(\underline{u}^s, \underline{\underline{\sigma}}^s)$

satisfying BCs



- Instantaneous

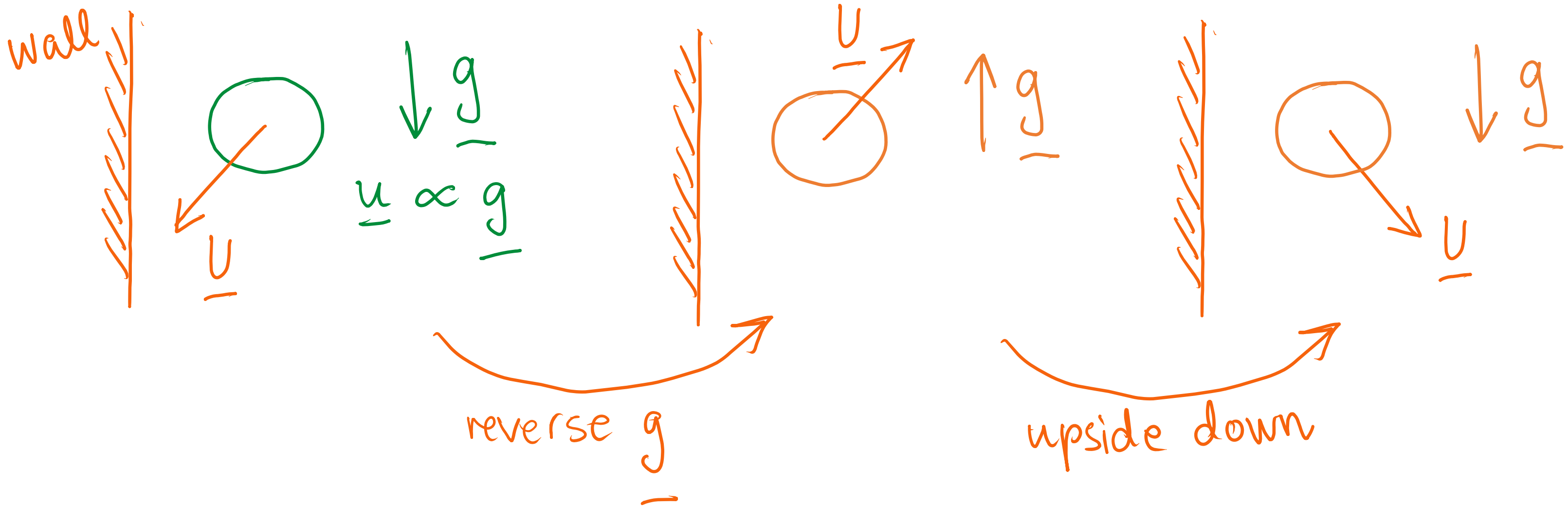
- Reversible in time



Scallop theorem

- Linear (in forcing and BCs)

- Reversible in space



Properties of Stokes flow (ctd.)

- Harmonic solutions $$0 \quad = \quad -\nabla p + \mu \nabla^2 \mathbf{u} \quad (*)$$

$\nabla \cdot (*)$ $\quad \nabla^2 p = 0 \quad$ by $\nabla \cdot \underline{u} = \underline{0}$

$\nabla \times (*)$ $\quad \nabla^2 \underline{\omega} = 0 \quad$ by $\underline{\omega} = \nabla \times \underline{u}$

$\nabla^2 (*)$ $\quad \nabla^2 \nabla^2 \underline{u} = \underline{0}$

Theorems of Stokes Flow

Lemma. Let $(\mathbf{u}^S(\mathbf{x}), \sigma^S(\mathbf{x}))$ be the Stokes flow on volume V (with body force $\mathbf{f} = 0$) and $\mathbf{u}(\mathbf{x})$ be any incompressible field on V. Then we have

$$\int_V 2\mu e_{ij}^S e_{ij}\, dV = \int_{\partial V} \sigma_{ij}^S u_i n_j\, dS.$$

Recall $\sigma_{ij}^S = -p^S\delta_{ij} + 2\mu e_{ij}^S$

Proof

$$\text{LHS} = \int_V (\sigma_{ij}^S + p^S\delta_{ij}) e_{ij} = \int_V \sigma_{ij}^S e_{ij}\, dV + \int_V p^S \underbrace{\cancel{e_{ii}}}_{0 \text{ as } \nabla\cdot\underline{u}=0}\, dV$$

$$e_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) = \frac{\partial u_i}{\partial x_j} - \underbrace{\frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}\right)}_{}$$

antisymmetric $\Rightarrow \sigma_{ij}^S(\dots) = 0$

$$= \int_V \sigma_{ij}^S \frac{\partial u_i}{\partial x_j}\, dV$$

$$= \int_V \frac{\partial}{\partial x_j}(\sigma_{ij}^S u_i)\, dV \quad \text{as } \partial_j \sigma_{ij}^S = 0 \text{ (Stokes eqn)}$$

$$= \int_{\partial V} \sigma_{ij}^S u_i n_j\, dS \quad \text{(by Divergence Theorem)} \qquad \square$$

Theorems of Stokes Flow

Lemma. Let $(\mathbf{u}^S(\mathbf{x}), \sigma^S(\mathbf{x}))$ be the Stokes flow on volume V (with body force $\mathbf{f} = 0$) and $\mathbf{u}(\mathbf{x})$ be any incompressible field on V. Then we have

symmetric

$$\int_V 2\mu e_{ij}^S e_{ij} \; dV = \int_{\partial V} \sigma_{ij}^S u_i n_j \; dS.$$

Corollary (Lorentz reciprocal theorem). Let $(\mathbf{u}^1(\mathbf{x}), \sigma^1(\mathbf{x}))$ and $(\mathbf{u}^2(\mathbf{x}), \sigma^2(\mathbf{x}))$ be two Stokes flows on the same volume V, with different boundary conditions. Then

$$\int_{\partial V} \sigma_{ij}^1 u_i^2 n_j \; dS = \int_{\partial V} \sigma_{ij}^2 u_i^1 n_j \; dS.$$

Proof Apply Lemma with $(\underline{u}^S, \underline{u}) = (\underline{u}^1, \underline{u}^2)$

$\&$ $(\underline{u}^S, \underline{u}) = (\underline{u}^2, \underline{u}^1)$ $\square$

Theorems of Stokes Flow

Lemma. Let $(\mathbf{u}^S(\mathbf{x}), \sigma^S(\mathbf{x}))$ be the Stokes flow on volume V (with body force $\mathbf{f} = 0$) and $\mathbf{u}(\mathbf{x})$ be any incompressible field on V. Then we have

$$\int_V 2\mu e_{ij}^S e_{ij} \, dV = \int_{\partial V} \sigma_{ij}^S u_i n_j \, dS.$$

Theorem (Uniqueness). There is a unique Stokes flow $(\mathbf{u}^S(\mathbf{x}), \sigma^S(\mathbf{x}))$ on a volume V, for given boundary conditions $\mathbf{u}^S(\mathbf{x}) = \mathbf{U}(\mathbf{x})$ on ∂V.

Proof. Suppose there were two such flows, $\underline{u}^1, \underline{u}^2$. Define $\underline{u}^* = \underline{u}_1 - \underline{u}_2$.

Apply lemma to $\underline{u} = \underline{u}^*$, $\underline{u}^S = \underline{u}^*$ (note $\underline{u}^*$ also Stokes flow by linearity)

$$\int_V 2\mu \underbrace{e_{ij}^* e_{ij}^*}_{\geq 0} dV = \int_{\partial V} \sigma_{ij}^* \underbrace{u_i^*}_{=0 \text{ as } \underline{u}_1 = \underline{u}_2 = \underline{U} \text{ on } \partial V} n_j \, dS \Rightarrow \underline{\underline{e_{ij}^* = 0 \text{ on } V}} \Rightarrow \underline{u}^* \text{ is rigid body motion}$$

But $\underline{u}^* = \underline{0}$ on $\partial V \Rightarrow \underline{u}^* \equiv \underline{0}$

□

Final remarks

- Remember to attempt the exercises for this topic before the live session on

  2pm Thursday, 8 October

- Go on YouTube and search "*reversibility of Stokes flow*"
- If you find any typos/mistakes in the PDF notes, please email me at mt599@cam.ac.uk

Thank you for watching!