Lubrication theory

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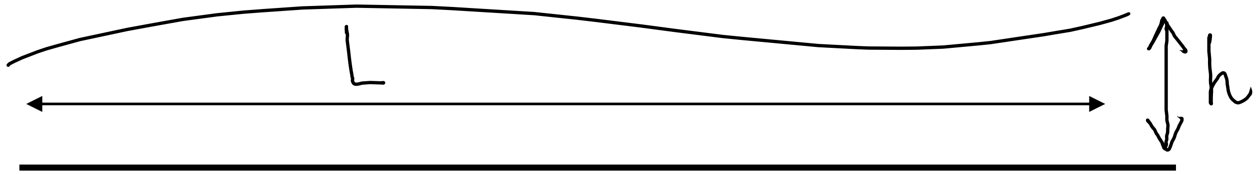
Part III Preparatory Workshop 2020

DAMTP, University of Cambridge

Long, narrow geometries



$$\underbrace{\rho\left(\cancel{\frac{\partial u}{\partial t}} + \underbrace{\cancel{u\frac{\partial u}{\partial x}} + \cancel{v\frac{\partial u}{\partial y}}}_{\sim U^2/L}\right)}_{\sim \rho U^2/L} = -\frac{\partial p}{\partial x} + \underbrace{\mu\left(\underbrace{\cancel{\frac{\partial^2 u}{\partial x^2}}}_{\frac{U}{L^2}} \ll \underbrace{\frac{\partial^2 u}{\partial y^2}}_{\frac{U}{h^2}}\right)}_{\sim \mu U/h^2}$$

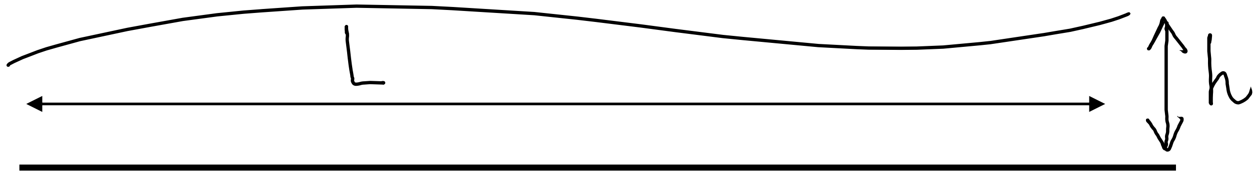
$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \Rightarrow \frac{U}{L} \sim \frac{V}{h}$$

$$\Rightarrow V \sim \frac{h}{L} U \ll U$$

$$\frac{\rho U^2}{L} \ll \frac{\mu U}{h^2} \iff \frac{\rho U h}{\mu} \cdot \frac{h}{L} \ll 1$$

$$\boxed{\begin{array}{l} ①\ h \ll L \\ ②\ \frac{h}{L}\,\mathrm{Re} \ll 1 \end{array}}$$

Long, narrow geometries



$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \qquad \frac{\partial p}{\partial x} = \mu \frac{\partial^2 u}{\partial y^2} \quad \Rightarrow \quad p \sim \frac{\mu U L}{h^2}$$

$$\underbrace{\cancel{\rho\left(\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y}\right)}}_{\sim \rho U V/L \;\sim\; \rho U^2 h/L^2} = \underbrace{-\frac{\partial p}{\partial y}}_{\sim \frac{\mu U L}{h^3}} + \underbrace{\mu\left(\cancel{\frac{\partial^2 v}{\partial x^2}} + \cancel{\frac{\partial^2 v}{\partial y^2}}\right)}_{\sim \mu U^2/L^2}$$

$\frac{h^3}{L^3} Re : 1$

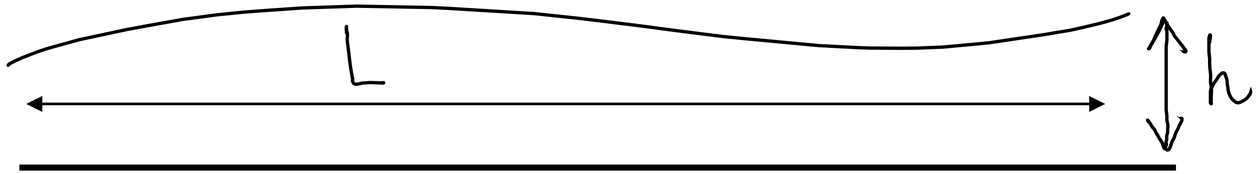
$\frac{L^3}{h^3} : 1$

$\Rightarrow \quad \frac{\partial p}{\partial y} = 0$

1. $h \ll L$
2. $\frac{h}{L} Re \ll 1$

Lubrication equations

$\downarrow \underline{f}$



$$\boxed{\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \qquad \frac{\partial p}{\partial x} = \mu \frac{\partial^2 u}{\partial y^2} + f_x, \qquad \frac{\partial p}{\partial y} = f_y}$$

Equations

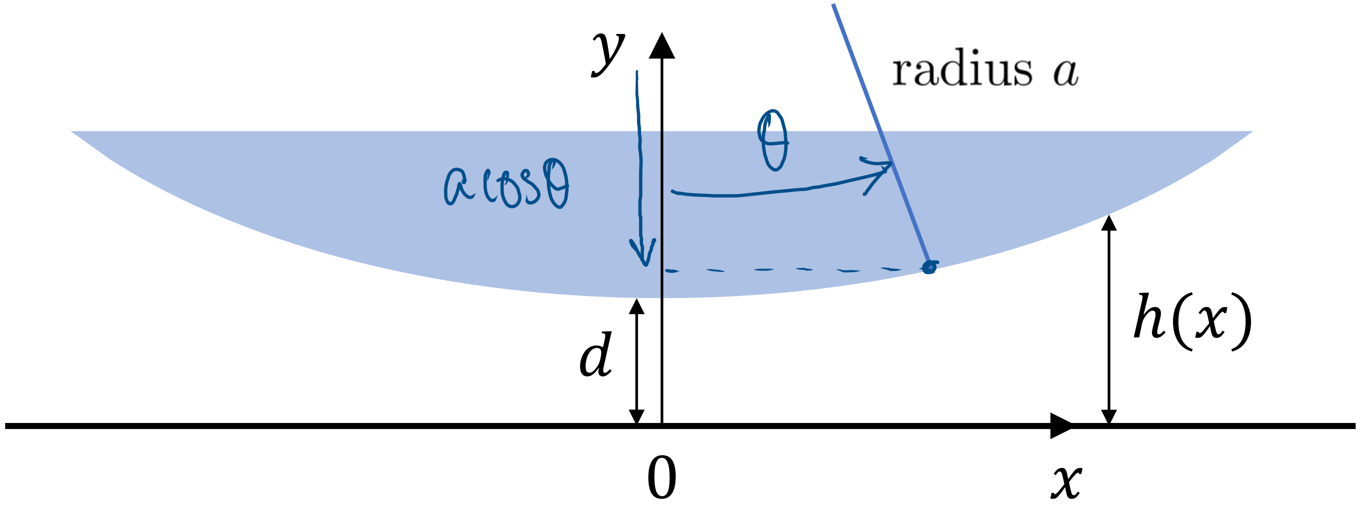
Assumptions

1. $h \ll L$
2. $\frac{h}{L} Re \ll 1$

How to solve a lubrication problem:

1. Describe the geometry.
2. Solve for (almost) unidirectional flow.
3. Apply mass conservation to close the system.
4. Calculate quantities of interest.

Ex: cylinder approaching wall



Hor. lengthscale L

$L = \sqrt{ad}$

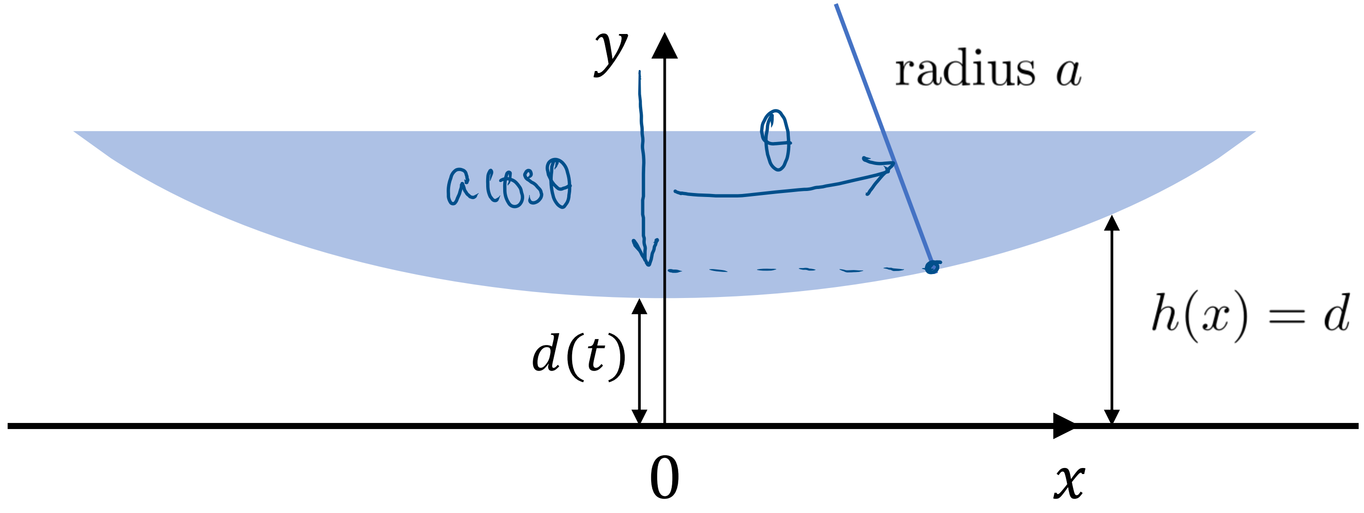
$\frac{d}{L} \sim \sqrt{\frac{d}{a}}$

1. Describe the geometry.

$$h(x) = d + a(1-\cos\theta) \approx d + \frac{a}{2}\theta^2 \quad (\theta \ll 1)$$

$$\theta \approx \sin\theta = \frac{x}{a} \Rightarrow h(x) \approx d + \frac{x^2}{2a} = d\left(1 + \frac{x^2}{2ad}\right)$$

Ex: cylinder approaching wall



$$h(x) = d\left(1 + \frac{x^2}{2ad}\right)$$

Boundary conditions

$\underline{y=0}$: $u = v = 0$

$\underline{y=h}$: $u = 0$

$v = \dot{d}$

2. Solve for (almost) unidirectional flow.

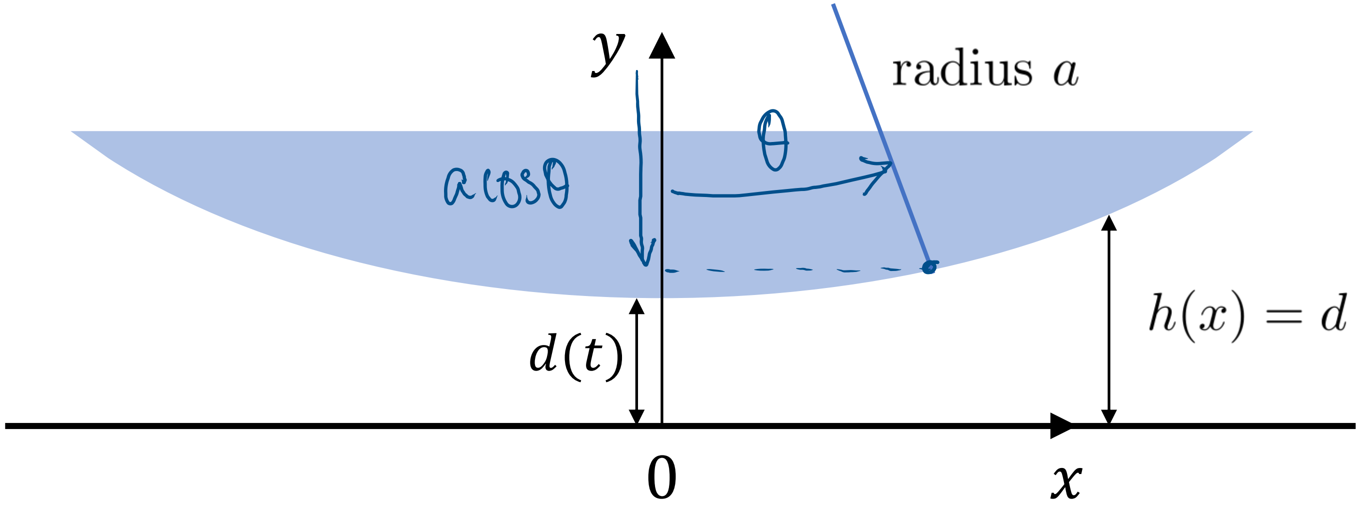
$$\frac{\partial p}{\partial y} = 0, \qquad \frac{\partial p}{\partial x} = \mu \frac{\partial^2 u}{\partial y^2}$$

$\Downarrow$ $\qquad\qquad$ $\Downarrow$

$p = p(x)$ $\qquad\qquad$ $u = \dfrac{1}{2\mu}\dfrac{\partial p}{\partial x}\, y(y-h)$

Q: What is $\dfrac{\partial p}{\partial x}$?

Ex: cylinder approaching wall



$$h(x) = d\left(1 + \frac{x^2}{2ad}\right)$$

Boundary conditions

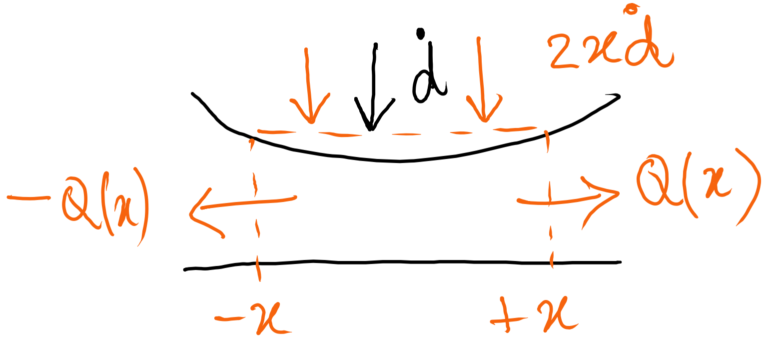
$\underline{y=0}$: $u = v = 0$

$\underline{y=h}$; $u = 0$

$v = \dot{d}$

$$\boxed{u = \frac{1}{2\mu}\frac{\partial p}{\partial x} y(y-h)}$$

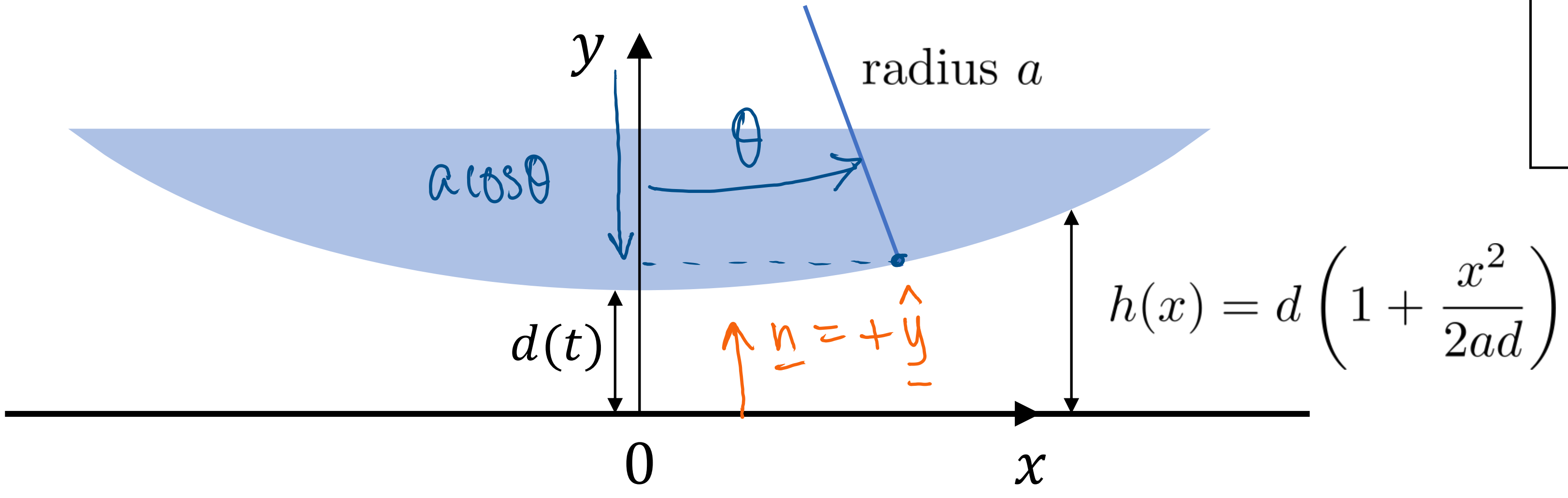
3. Apply mass conservation to close the system.



$2x\dot{d} = 2Q(x)$

$$\Rightarrow \quad Q(x) = \int_0^h u\,dy = -\frac{h^3}{12\mu}\frac{dp}{dx} = x\dot{d}$$

Ex: cylinder approaching wall



$\sim \dfrac{\mu U L}{h^2}$ $\quad \sim \dfrac{\mu U}{L}$

$\sigma_{yy} = -p + \cancel{\mu \dfrac{\partial v}{\partial y}}$

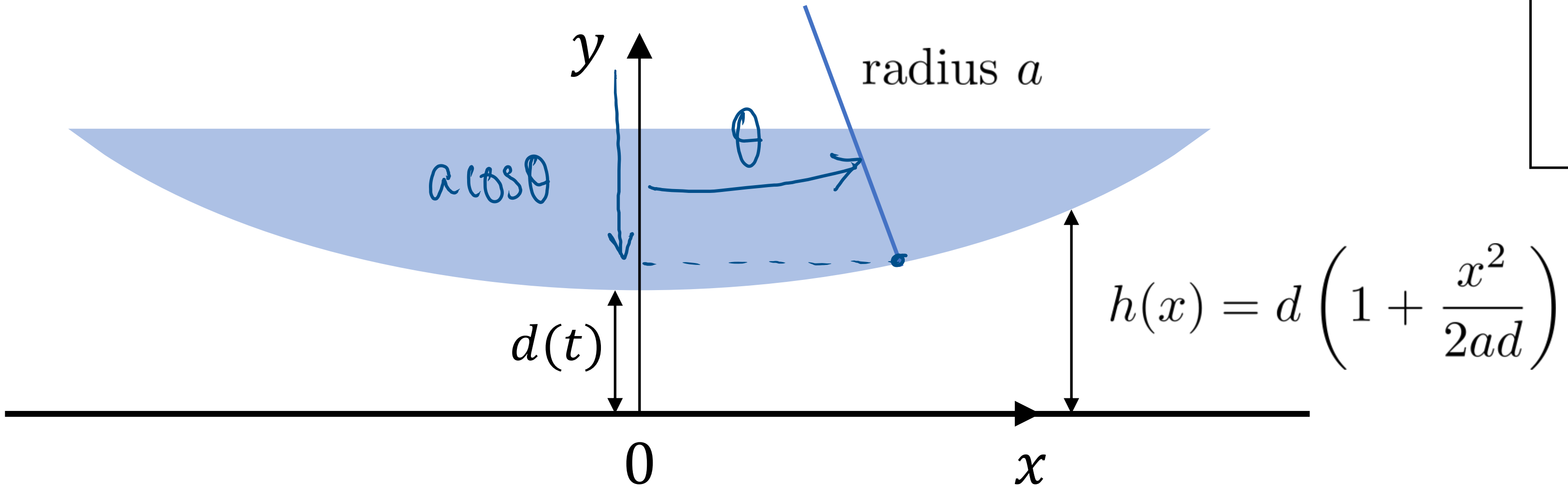
negligible

4. Calculate quantities of interest.

$$\frac{dp}{dx} = -\frac{12\mu\dot{d}}{d^3} \times \frac{x}{(1+x^2/2ad)^3} \qquad \Rightarrow p(x) = \frac{6\mu\dot{d}}{d^2(1+x^2/2ad)^2}$$

$-\underbrace{\text{Normal force}}_{\underline{F} = F\hat{\underline{y}}}$ on cylinder $=$ normal force on plate $= \displaystyle\int_{-\infty}^{+\infty} \sigma_{yy}\, dx = \int_{-\infty}^{+\infty} -p\, dx$

Ex: cylinder approaching wall



4. Calculate quantities of interest.

$$p(x) = \frac{6\mu\dot{d}}{d^2(1+x^2/2ad)^2}, \qquad F = \int_{-\infty}^{\infty} p \; dx = \cdots = 3\sqrt{2}\pi\mu\,\dot{d}\,(a/d)^{3/2}$$

Sedimentation under gravity $\Rightarrow F = -mg = \text{constant}$

$\Rightarrow \dot{d} \sim -d^{3/2} \Rightarrow \boxed{d \sim t^{-2}}$

Final remarks

- Remember to attempt the exercises for this topic before the live session on

  2pm Thursday, 8 October

- If you find any typos/mistakes in the PDF notes, please email me at mt599@cam.ac.uk

Thank you for watching!