Boundary layers

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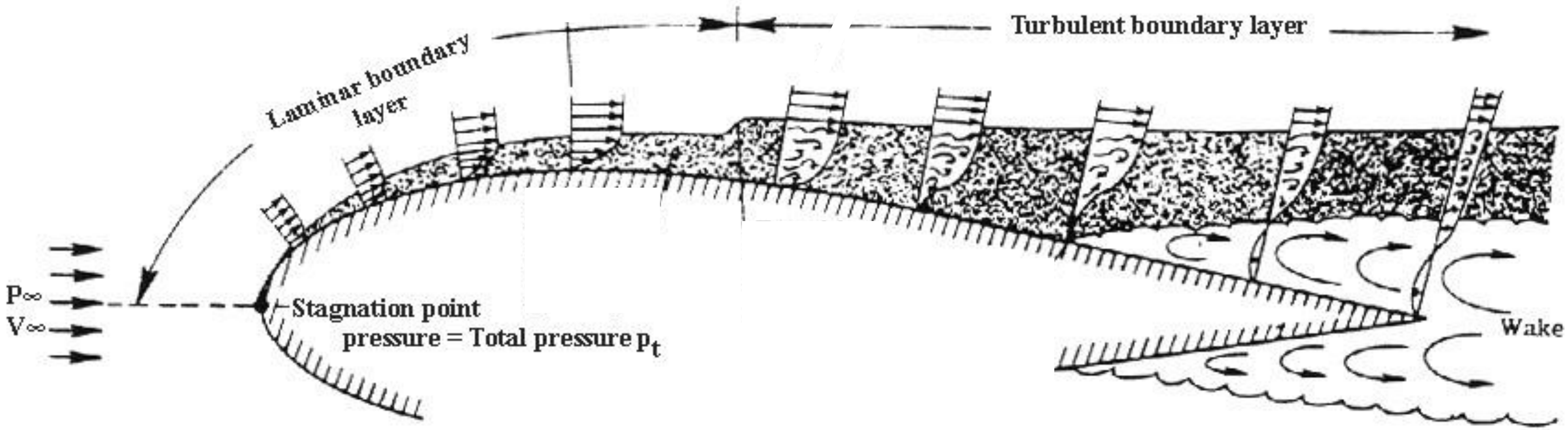
Motivation

Bcs : $\underline{u} \cdot \underline{n} = 0$

but not $\underline{u} \times \underline{n} = \underline{0}$

(no-slip)

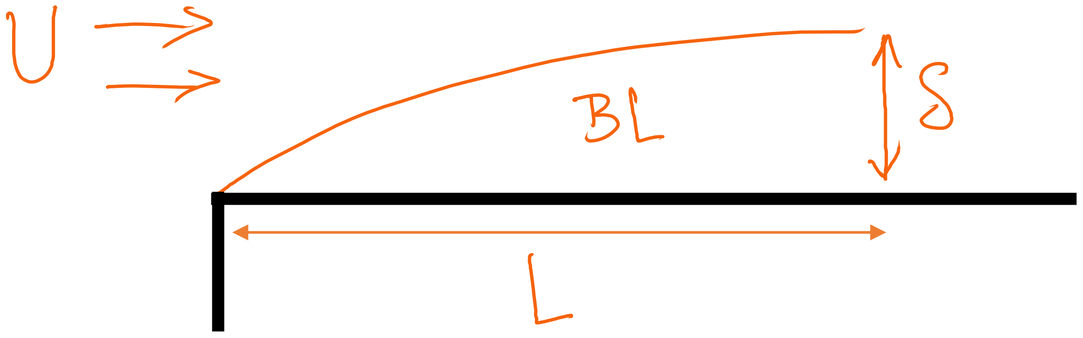
Outside BL : $\underline{u} = \nabla \phi$



include viscosity in BL

From: https://aerospaceengineeringblog.com/boundary-layer-separation-and-pressure-drag/
[accessed 26/09/2020] Original source unclear.

Diffusion of vorticity



Kinematic viscosity ν $\qquad \delta \sim \sqrt{\nu t} \sim \sqrt{\frac{\nu L}{U}} \Rightarrow \frac{\delta}{L} \sim \sqrt{\frac{\nu}{LU}} = Re^{-1/2}$

Mass conservation: $\qquad \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \Rightarrow \frac{U}{L} \sim \frac{V}{\delta} \Rightarrow V \sim Re^{-1/2} U$

Boundary layer equations (2D)

$$x: \quad \rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right) = -\frac{\partial p}{\partial x} + \mu\left(\cancel{\frac{\partial^2 u}{\partial x^2}} + \frac{\partial^2 u}{\partial y^2}\right)$$

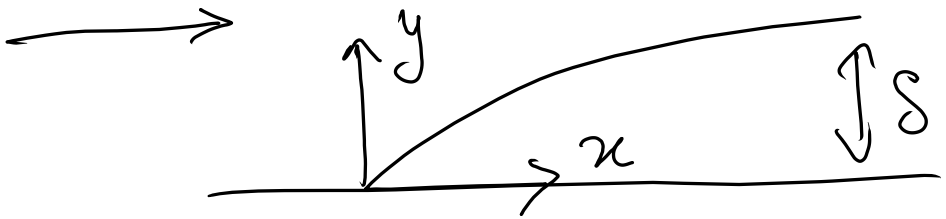
same order as $\frac{U}{L} \sim \frac{V}{\delta}$

small as $\frac{1}{L^2} \ll \frac{1}{\delta^2}$

Same order as $\frac{\rho U^2}{L} \sim \frac{\mu U}{\delta^2} \iff \frac{\delta}{L} \sim \left(\frac{UL}{\nu}\right)^{-1/2}$

Outside BL: $\quad \rho\left(\frac{\partial U}{\partial t} + U\frac{\partial U}{\partial x}\right) = -\frac{\partial p}{\partial x} \quad \Rightarrow \quad p \sim \rho U^2$

$U(x,t)$



Boundary layer equations (2D)

Recap : $\frac{\delta}{L} \sim Re^{-1/2}$, $V \sim \frac{\delta}{L} U$, $p \sim \rho U^2$

Same order

$$y: \quad \rho\left(\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y}\right) = -\frac{\partial p}{\partial y} + \mu\left(\cancel{\frac{\partial^2 v}{\partial x^2}} + \frac{\partial^2 v}{\partial y^2}\right)$$

same order $\sim \frac{\rho V^2}{\delta}$

small

$\sim \frac{\rho U^2}{\delta} \gg \frac{\rho V^2}{\delta}$ as $V \sim Re^{-1/2} U$

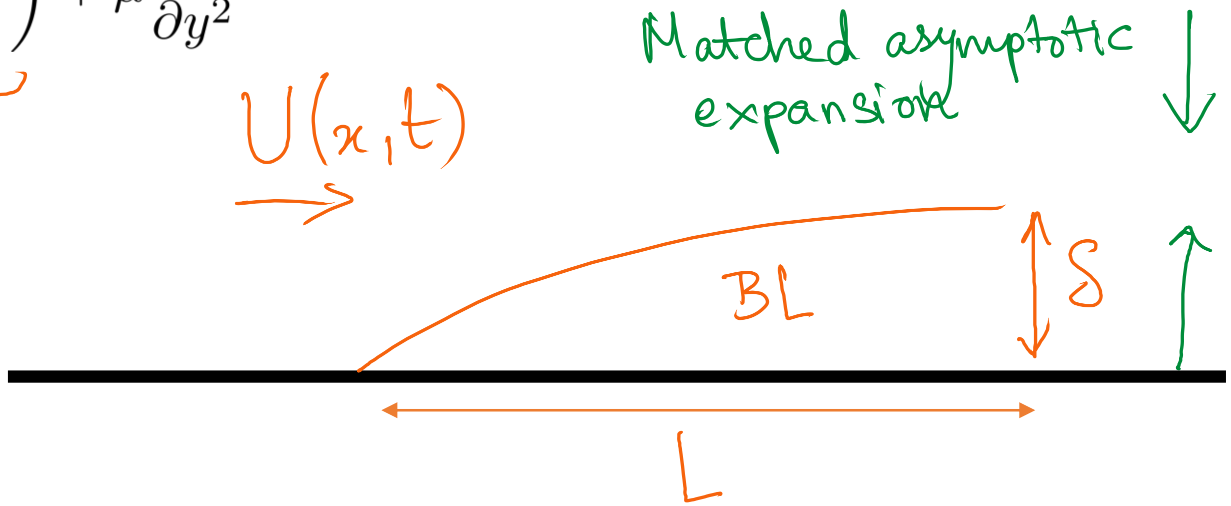
$$\boxed{\Rightarrow \quad \frac{\partial p}{\partial y} = 0}$$

$$-\frac{\partial p_{\mathrm{inBL}}}{\partial x} = -\frac{\partial p_{\mathrm{outBL}}}{\partial x} = \rho\left(\frac{\partial U}{\partial t} + U\frac{\partial U}{\partial x}\right)$$

Boundary layer equations (2D)

$$\rho\left(\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}\right)=\underbrace{\rho\left(\frac{\partial U}{\partial t}+U\frac{\partial U}{\partial x}\right)}_{-\frac{\partial p}{\partial x}\ \text{from outside}}+\mu\frac{\partial u}{\partial y^2}$$

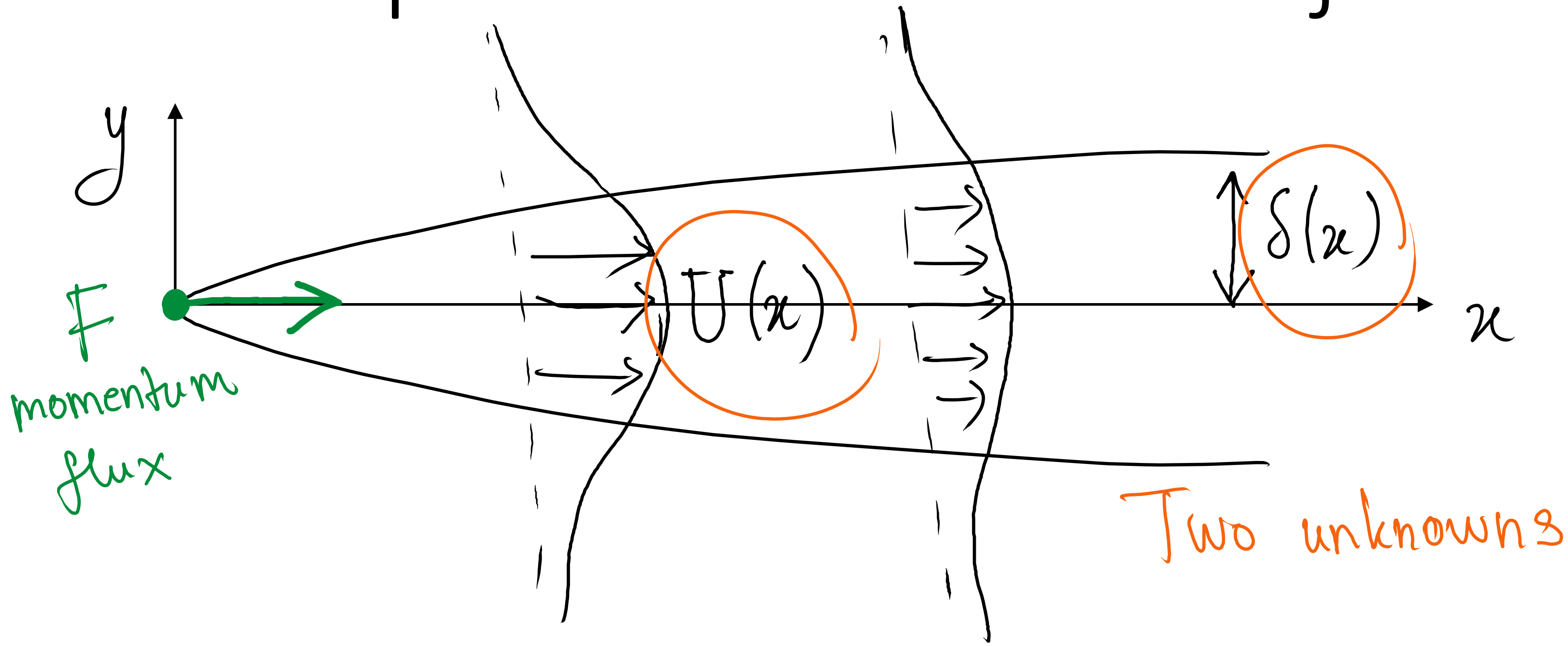
$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0$$



Boundary conditions:

$$\begin{aligned} u,v &= 0 \quad \text{on } y=0 \\ u &\to U \quad \text{as } \frac{y}{\delta}\to\infty \end{aligned}$$

Example: 2D momentum jet



Steady $\Rightarrow$ $\frac{\partial u}{\partial t} = 0$

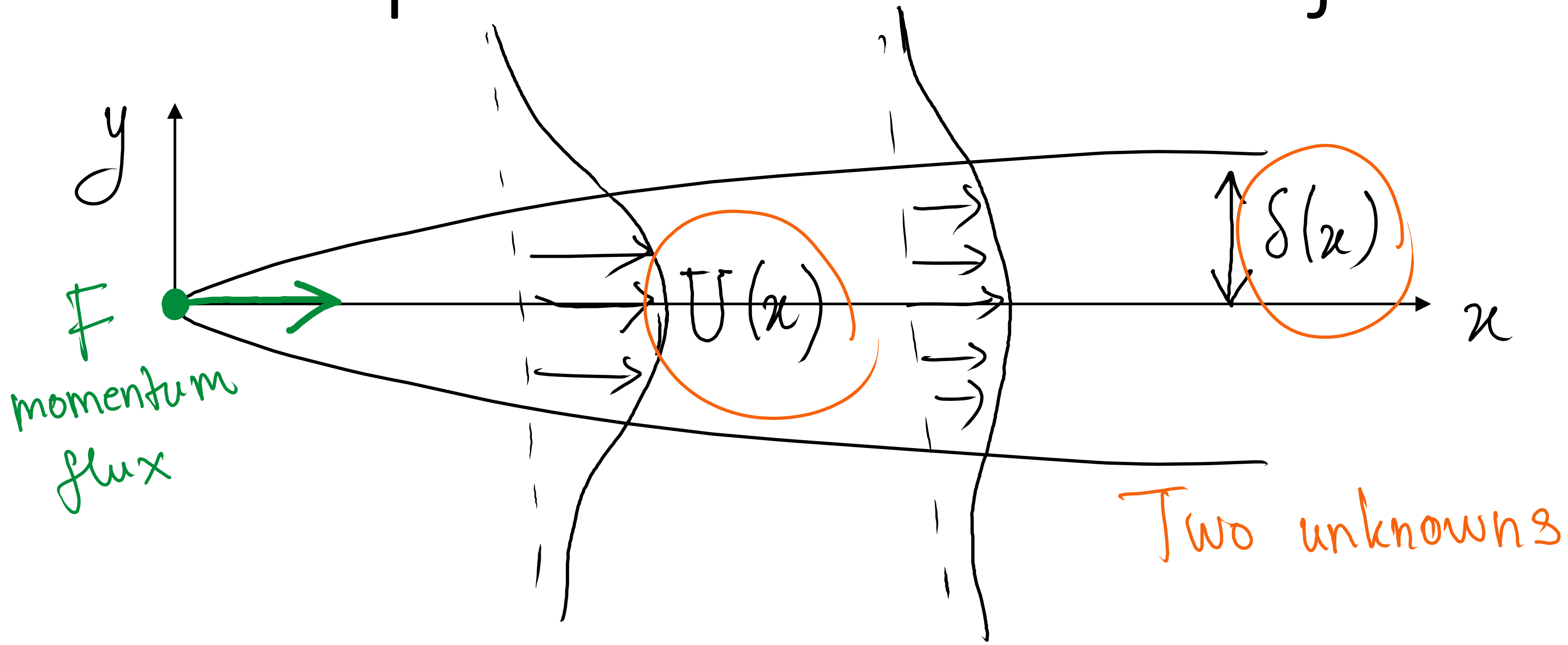
No flow at infinity $\Rightarrow$ $\left.\frac{\partial p}{\partial x}\right|_{out} = 0$

$$\rho\left(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right) = \mu\frac{\partial u}{\partial y^2}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

$$\begin{aligned} v &= 0 \quad \text{on } y = 0 \\ u &\to 0 \quad \text{as } \frac{y}{\delta} \to \infty \end{aligned}$$

Example: 2D momentum jet



Momentum balance : $\rho \dfrac{U^2(x)}{x} \sim \mu \dfrac{U(x)}{\delta(x)^2}$

Constant momentum flux : $\rho U^2(x)\,\delta(x) \sim F$

$$\Rightarrow \quad \delta(x) = \left(\frac{\rho \nu^2 x^2}{F}\right)^{\frac{1}{3}}, \qquad U(x) = \left(\frac{F^2}{\rho^2 \nu x}\right)^{\frac{1}{3}}$$

$$\rho\left(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right) = \mu\frac{\partial u}{\partial y^2}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

$$\begin{aligned} v &= 0 \quad \text{on } y = 0 \\ u &\to 0 \quad \text{as } \frac{y}{\delta} \to \infty \end{aligned}$$

$\delta(x) \sim x^{2/3}, \quad U(x) \sim x^{-1/3}$

Why constant momentum flux?

$$F(x) = \int_{-\infty}^{+\infty} \rho u^2 \, dy$$

BL eqn

$$\Rightarrow \frac{dF}{dx} = \int_{-\infty}^{+\infty} 2\rho u \frac{\partial u}{\partial x} \, dy \overset{\downarrow}{=} 2 \int_{-\infty}^{+\infty} \left(\mu \frac{\partial^2 u}{\partial y^2} - \rho v \frac{\partial u}{\partial y} \right) dy$$

$$= 2\mu \underbrace{\cancel{\left[\frac{\partial u}{\partial y} \right]_{-\infty}^{+\infty}}}_{0} - 2\rho \underbrace{\cancel{\left[vu \right]_{-\infty}^{+\infty}}}_{0} + 2\rho \int_{-\infty}^{+\infty} u \underbrace{\frac{\partial v}{\partial y}}_{= -\frac{\partial u}{\partial x} \text{ from } \nabla \cdot \underline{u} = 0} dy$$

as $u \to 0$ as $y \to \pm\infty$ same

$$\Rightarrow \frac{dF}{dx} = -\frac{dF}{dx} = 0$$

$\delta \sim x^{2/3}$

$\Rightarrow \frac{\partial \eta}{\partial x} = -\frac{2}{3}\frac{\eta}{x}$

② Similarity solution

$$\delta(x) = \left(\frac{\rho\nu^2 x^2}{F}\right)^{1/3}, \quad U(x) = \left(\frac{F^2}{\rho^2 \nu x}\right)^{1/3}$$

① Streamfunction ψ s.t. $u = \frac{\partial \psi}{\partial y}$, $v = -\frac{\partial \psi}{\partial x}$

$$\psi(x,y) = \Psi(x,\eta) = \underbrace{U(x)\delta(x)}_{\text{s.t. } \frac{\partial \psi}{\partial y} \sim U(x) \text{ as required}} f(\eta), \quad \eta = \frac{y}{\delta(x)}$$

Sub into ⊛ : $\frac{\partial}{\partial y} = \frac{1}{\delta}\frac{\partial}{\partial \eta}$, $\frac{\partial}{\partial x} = \frac{\partial}{\partial x} - \frac{2}{3}\frac{\eta}{x}\frac{\partial}{\partial \eta}$

$\Rightarrow u = \frac{\partial \psi}{\partial y} = U(x) f'(\eta)$, $v = \frac{U\delta}{x}\left(-\frac{1}{3}f + \frac{2}{3}\eta f'\right)$

$$\rho\left(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right) = \mu\frac{\partial u}{\partial y^2} \quad ⊛$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \quad ✓$$

$$\begin{aligned} v &= 0 \quad \text{on } y = 0 \\ u &\to 0 \quad \text{as } \frac{y}{\delta} \to \infty \end{aligned}$$

Similarity solution

$$\delta(x) = \left(\frac{\rho\nu^2 x^2}{F}\right)^{1/3}, \quad U(x) = \left(\frac{F^2}{\rho^2 \nu x}\right)^{1/3}$$

$$\psi(x,y) = \Psi(x,\eta) = U(x)\delta(x)f(\eta), \quad \eta = \frac{y}{\delta(x)}$$

$$\rho\left(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right) = \mu\frac{\partial u}{\partial y^2} \qquad (*)$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

$$\begin{array}{llll} v & = & 0 & \text{on } y = 0 \qquad (1) \\ u & \to & 0 & \text{as } \dfrac{y}{\delta} \to \infty \qquad (2) \end{array}$$

$(*) \Rightarrow -\frac{1}{3}f'^2 - \frac{1}{3}ff'' = f'''$

(1) $\Rightarrow$ $f(0) = 0$

(2) $\Rightarrow$ $f' \to 0$ as $\eta \to \pm\infty$

(3) : const. mom flux $\Rightarrow$ $\int_{-\infty}^{+\infty} (f')^2 \, d\eta = 1$

Mass flux : $\int_{-\infty}^{+\infty} \rho U \, dy \sim \rho U(x)\delta(x) \sim x^{1/3}$

Final remarks

- Remember to attempt the exercises for this topic before the live session on

2pm Thursday, 8 October

- If you find any typos/mistakes in the PDF notes, please email me at mt599@cam.ac.uk

Thank you for watching!