Internal gravity waves

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Part III Preparatory Workshop 2020

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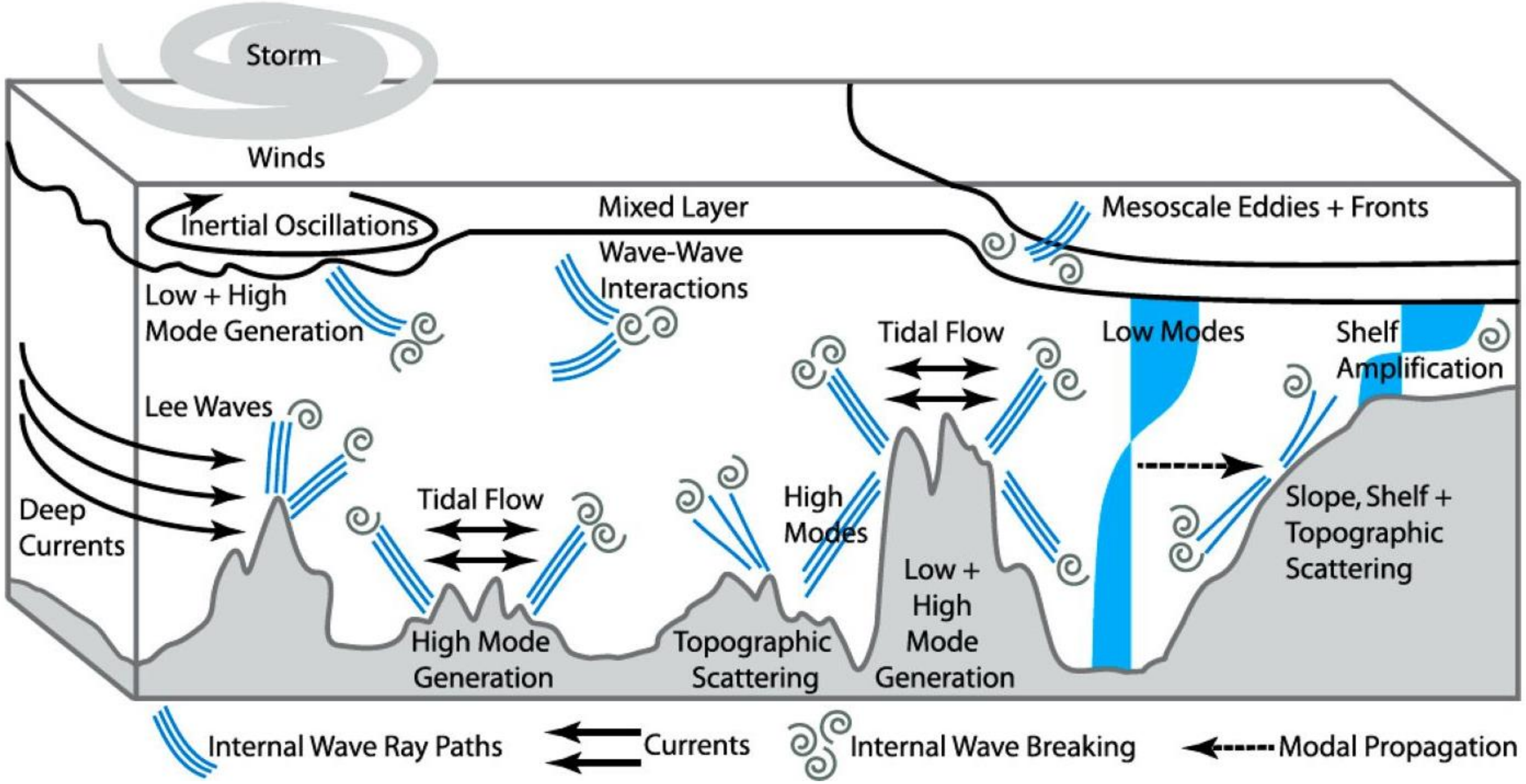
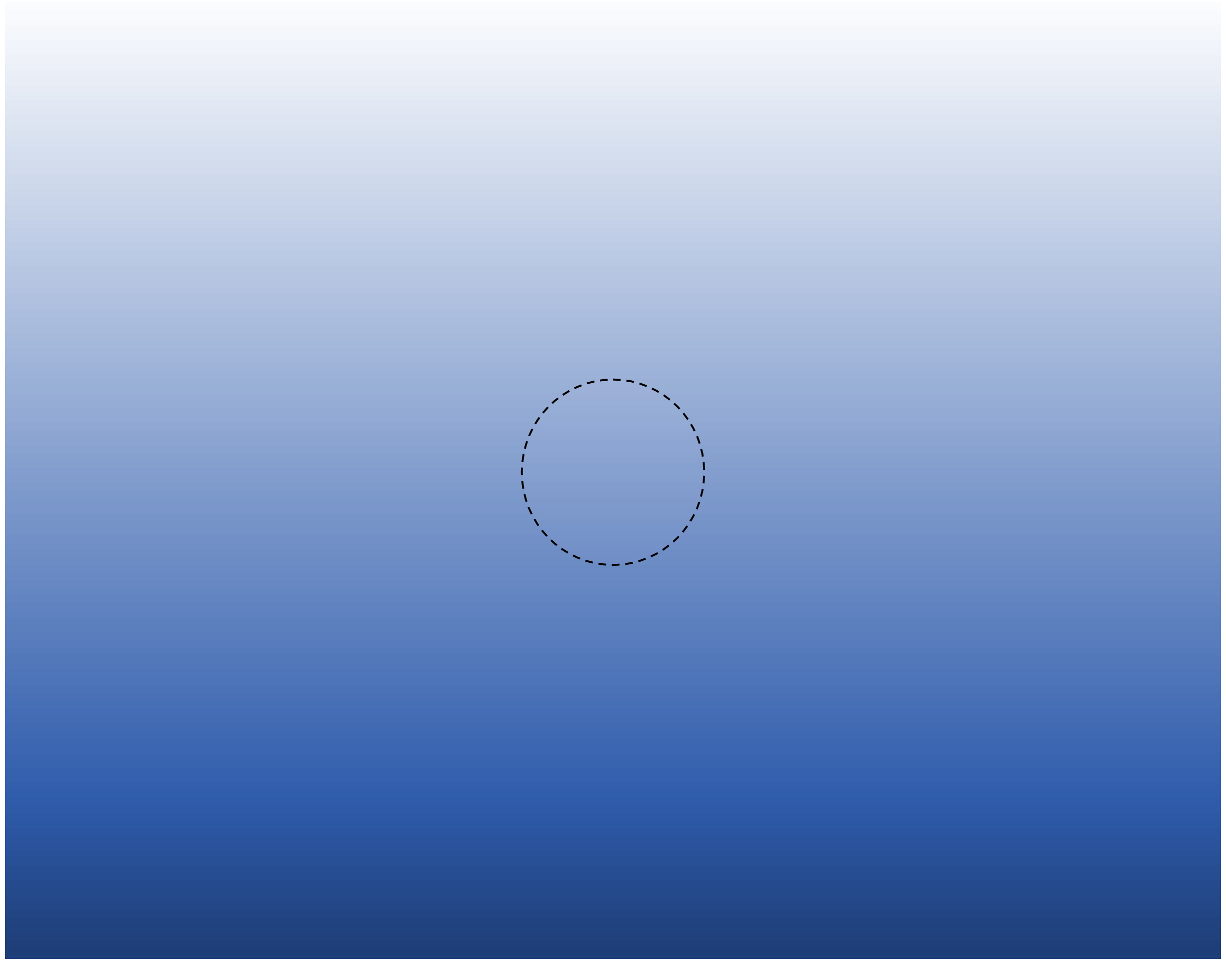
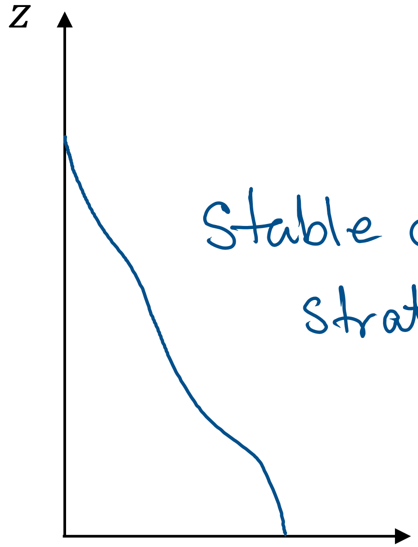
Motivation



MacKinnon, J. A. et al. (2017) *Bull. Am. Meteor. Soc*. **98**, 2429–2454

Physical mechanism





Stable density stratification

$$\frac{d\rho_0}{dz} < 0$$

Governing equations

$$
\begin{aligned}
\nabla \cdot \mathbf{u} &= 0 \\
\rho \left(\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} \right) &= -\nabla p + \rho \mathbf{g} \\
\frac{\partial \rho}{\partial t} + \mathbf{u} \cdot \nabla \rho &= 0
\end{aligned}
$$

Base state: $\mathbf{u} = 0, \quad \rho = \rho_0(z), \quad p = p_0 - \int \rho_0(z) g \, dz$

Linear perturbations: $\underline{u} = \tilde{\underline{u}}$, $\rho = \rho_0 + \tilde{\rho}$

$p = p_{base} + \tilde{p}$

Linearised equations:

$$
\begin{aligned}
\nabla \cdot \tilde{\mathbf{u}} &= 0 && (1) \\
\rho_0 \frac{\partial \tilde{\mathbf{u}}}{\partial t} &= -\nabla \tilde{p} + \tilde{\rho} \mathbf{g} && (2) \\
\frac{\partial \tilde{\rho}}{\partial t} + \tilde{w} \frac{d\rho_0}{dz} &= 0 && (3)
\end{aligned}
$$

Boussinesq approximation ?

$$\nabla \cdot \tilde{\mathbf{u}} = 0 \qquad (1)$$

$$\rho_0 \frac{\partial \tilde{\mathbf{u}}}{\partial t} = -\nabla \tilde{p} + \tilde{\rho}\mathbf{g} \qquad (2)$$

Eliminate!

$$\frac{\partial \tilde{\rho}}{\partial t} + \tilde{w}\frac{d\rho_0}{dz} = 0 \qquad (3)$$

Say ρ_0 and ρ_0' vary slowly on wavelength of IGW, so treat as constants.

$$\nabla \times \frac{\partial}{\partial t}(2) \Rightarrow \quad \rho_0 \frac{\partial^2}{\partial t^2} \nabla \times \underline{u} = -\frac{\partial}{\partial t} \underbrace{\cancel{\nabla \times \nabla \tilde{p}}}_{0} + \underbrace{\frac{\partial}{\partial t}\left(\nabla \times \tilde{\rho}\underbrace{\underline{g}}_{\text{constant}}\right)}_{-\underline{g} \times \frac{\partial}{\partial t}\nabla\tilde{\rho}}$$

By (3) $\Rightarrow + \underline{g} \times \nabla\left(w \frac{d\rho_0}{dz}\right)$

$$\boxed{\rho_0 \frac{\partial^2}{\partial t^2} \nabla \times \tilde{\mathbf{u}} = \frac{d\rho_0}{dz}\mathbf{g} \times \nabla \tilde{w}}$$

$$\nabla \times \left(\frac{\partial^2}{\partial t^2} \nabla \times \tilde{\mathbf{u}} = \frac{1}{\rho_0} \frac{d\rho_0}{dz} \mathbf{g} \times \nabla \tilde{w} \right)$$

$$\frac{\partial^2}{\partial t^2} \nabla \times (\nabla \times \tilde{\underline{u}}) = \frac{1}{\rho_0} \frac{d\rho_0}{dz} \nabla \times (\underline{g} \times \nabla \tilde{w})$$

$$\Rightarrow \frac{\partial^2}{\partial t^2} \underbrace{(-\nabla^2 \tilde{\underline{u}})}_{\text{using } \nabla \cdot \tilde{\underline{u}} = 0} = \frac{1}{\rho_0} \frac{d\rho_0}{dz} \left(\underline{g} \, \nabla^2 \tilde{w} - (\underline{g} \cdot \nabla) \nabla w \right)$$

$$\underline{e}_z \cdot \left(\underline{g} \nabla^2 - (\underline{g} \cdot \nabla) \nabla \right)$$

$$= -g \nabla^2 + g \frac{\partial^2}{\partial z^2}$$

z- component $\Rightarrow$ $$\left(\frac{\partial^2}{\partial t^2} \nabla^2 - N^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) \right) \tilde{w} = 0$$

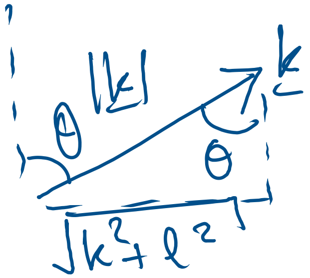
Brunt- Väisälä frequency $N^2 = -\frac{g}{\rho_0} \frac{d\rho_0}{dz} > 0$

Dispersion relation

$$\left(\frac{\partial^2}{\partial t^2}\nabla^2 - N^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\right)\tilde{w} = 0$$

Plane wave solutions : $\tilde{\underline{u}}, \tilde{p}, \tilde{\rho} \propto e^{i(\underline{k}\cdot\underline{x} - \omega t)}$

$$\Rightarrow \omega^2(-|\underline{k}|^2) - N^2(-k^2 - l^2) = 0 \qquad \underline{k} = (k, l, m)$$



$$\boxed{\omega^2 = \frac{N^2(k^2+l^2)}{k^2+l^2+m^2}} = N^2 \sin^2\theta$$

$$\Rightarrow \omega = \pm N \sin\theta$$

$$|\omega| < N = -\frac{g}{\rho_0}\frac{d\rho_0}{dz}$$

Energy & crests

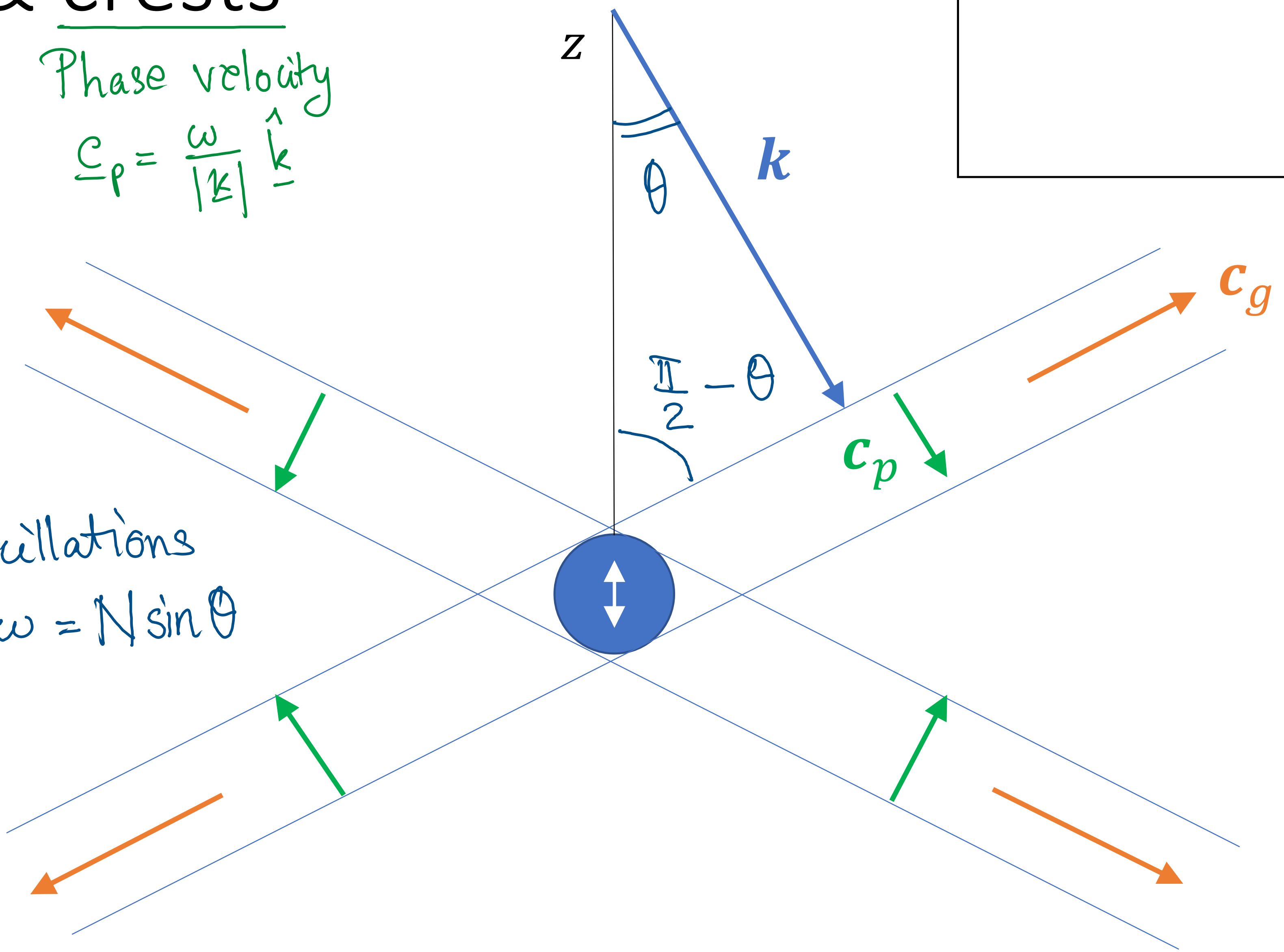
Group velocity

$$c_g = \frac{\partial \omega}{\partial \mathbf{k}}$$

$$(c_g)_i = \frac{\partial \omega}{\partial k_i}$$

Phase velocity

$$c_p = \frac{\omega}{|\mathbf{k}|}\hat{\mathbf{k}}$$



Oscillations

freq $\omega = N \sin\theta$

Final remarks

- Remember to attempt the exercises for this topic before the live session on

  2pm Thursday, 8 October

- Look inside the PDF notes for a link to a video of *internal gravity waves generated in a water tank*.
- If you find any typos/mistakes in the PDF notes, please email me at mt599@cam.ac.uk

Thank you for watching!